

A Conjecture on Zero-sum 3-magic Labeling of 5-regular Graphs [†]

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Abstract

In this paper, we obtained that every 5-regular graph admits a zero-sum 3-magic labeling, which give an affirmative answer to a conjecture proposed by Saieed Akbari, Farhad Rahmati and Sanaz Zare in *Electron. J. Combin.*.

Key Words: zero-sum magic labeling; degree sequence; 1-factor

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1. Introduction

Graph considered here are all finite and undirected with vertex set $V(G)$ and edge set $E(G)$. A *multigraph* is a graph with multiple edges. If every vertex in a graph has the same degree r then this graph is referred to as a *r-regular* graph. A matching M in G is a set of independent edges, and $|M|$ denotes the number of edges in M . A *factor* of a graph G is a spanning subgraph of G . A *k-factor* of G is a factor of G that is *k-regular*. Thus a 1-*factor* of G is a matching that saturates all vertices of G , and is called a *perfect matching* of G . A mapping $l : E(G) \rightarrow A$, where A is an abelian group which written additively, is called a *labeling* of the graph G . Given a labeling l of the graph G , the symbol $s(v)$, which represents the sum of the labels of edges incident with v , is defined to be $s(v) = \sum_{uv \in E(G)} l(uv)$, where $v \in V(G)$. For every positive integer $h \geq 2$, a graph G is said to be *zero-sum h-magic* if there is an edge labeling from $E(G)$ into $\mathbb{Z}_h \setminus \{0\}$ such that $s(v) = 0$ for every vertex $v \in V(G)$. The *null set* of a graph G , denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that G admits a zero-sum h-magic labeling.

Recently, Saieed Akbari, Farhad Rahmati and Sanaz Zare [1] obtained the following interesting results about magic labeling of regular graphs.

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Theorem 1.1 ^[1] *Let G be an r -regular graph ($r \geq 3$, $r \neq 5$). If r is even, then $N(G) = \mathbb{N}$, otherwise $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$. Furthermore, if r ($r \neq 5$) is odd and G is a 2-edge connected r -regular graph, then $N(G) = \mathbb{N} \setminus \{2\}$.*

They also proposed the following conjecture in [1].

Conjecture *Every 5-regular graph admits a zero-sum 3-magic labeling.*

In this paper, we give an affirmative answer to this conjecture. The following lemma is essential in the proof of the conjecture.

Lemma 1.1 ^[2] *Let G be a graph of even order with degree sequence $d=(d_1, d_2, \dots, d_n)$. If $\tilde{d}=(d_1-1, d_2-1, \dots, d_n-1)$ is also a degree sequence of some graph, then G has a 1-factor.*

More information and related references concerning magic labeling of graphs can be seen in [1].

2. Main Results

In this section, we will give a proof of the Conjecture.

If a graph G has vertices v_1, v_2, \dots, v_n , the sequence $d=(d_1, d_2, \dots, d_n)$ is called the *degree sequence* of G , where $d_i = d(v_i)$ for $i = 1, 2, \dots, n$. A nonincreasing and nonnegative integer sequence $d=(d_1, d_2, \dots, d_n)$ is *graphical* if there is a simple graph with degree sequence d . It is obvious that the conditions $d_i \leq n-1$ for all i , and $\sum_{i=1}^n d_i$ being even are necessary for a sequence to be graphical. Firstly, the following lemma will be obtained.

Lemma 2.1 *Let n be a positive even number, and $d=(d_1, d_2, \dots, d_n)$ be a sequence of nonnegative integers. If $d_1 = d_2 = \dots = d_n = 5$ and $n \geq 6$, or $d_1 = d_2 = \dots = d_n = 4$ and $n \geq 6$, then d is graphical.*

Proof For convenience, we let G_n denote the corresponding graph related to the sequence $d=(d_1, d_2, \dots, d_n)$.

Firstly, we prove that if $d_1 = d_2 = \dots = d_n = 5$ and $n \geq 6$ then d is graphical. The proof is by induction on n . If $n = 6$, then it is a obvious result since the complete graph K_6 being the graph G_6 with degree sequence $(5, 5, 5, 5, 5, 5)$. When $n = 8$, the corresponding graph G_8 is obtained from G_6 through the following construction. Let $V(G_6)=\{v_1, v_2, \dots, v_6\}$. Firstly, we add two new vertices v_7 and v_8 to G_6 , and add an edge connecting v_7 and v_8 . Secondly, we select, in G_6 , two different matchings M_1 and M_2 with $M_1 \cap M_2 = \emptyset$ and $|M_1|=|M_2|=2$. Deleting the four edges in $M_1 \cup M_2$ from G_6 , and connecting the four vertices in M_1 to v_7 , the other four vertices in M_2 to v_8 , we get the graph G_8 with

degree sequence $(5, 5, 5, 5, 5, 5, 5, 5)$. Now, suppose that $n = 2(k + 1) \geq 10$. By induction hypothesis the $2k$ -elements sequence $(5, 5, \dots, 5)$ is graphical and the corresponding graph is G_{2k} . So the graph $G_{2(k+1)}$ can be obtained from G_{2k} through the same procedure as that of G_6 to G_8 , and the proof is complete.

As for the case $d_1 = d_2 = \dots = d_n = 4$ and $n \geq 6$, we also through the induction on n . If $n = 6$, then it is an easy work to find a 4-regular graph G_6 with degree sequence $(4, 4, 4, 4, 4, 4)$. When $n = 8$, the corresponding graph G_8 is obtained from G_6 through the following operation. Let $V(G_6) = \{v_1, v_2, \dots, v_6\}$. Firstly, we add two new vertices v_7 and v_8 to G_6 , and select, in G_6 , two different matchings M_1 and M_2 with $M_1 \cap M_2 = \emptyset$ and $|M_1| = |M_2| = 2$. Deleting the four edges in $M_1 \cup M_2$, and connecting the four vertices in M_1 to v_7 , the other four vertices in M_2 to v_8 , we get the graph G_8 with degree sequence $(4, 4, 4, 4, 4, 4, 4, 4)$. Now, suppose that $n = 2(k + 1) \geq 10$. By induction hypothesis the $2k$ -elements sequence $(4, 4, \dots, 4)$ is graphical and the corresponding graph is G_{2k} . So the graph $G_{2(k+1)}$ can be obtained from G_{2k} through the same procedure as that of G_6 to G_8 , and the proof is complete. \square

Theorem 2.1 *Every 5-regular graph admits a zero-sum 3-magic labeling.*

Proof It is obvious that every 5-regular graph G is of even order since $2 \cdot E(G) = 5 \cdot V(G)$. For $|V(G)| < 6$, the correctness of the theorem is easily to verify. When $|V(G)| \geq 6$, according to the Lemma 1.1 and Lemma 2.1 we can get that every 5-regular graph G contains a 1-factor. So, labeling the edges in the 1-factor with 2 ($\in \mathbb{Z}_3 \setminus \{0\}$) and the remaining edges with 1 ($\in \mathbb{Z}_3 \setminus \{0\}$), we will get a zero-sum 3-magic labeling of the 5-regular graph. \square

The following theorem can be easily deduced from the Theorem 1.1 and Theorem 2.1.

Theorem 2.2 *Let G be an r -regular graph with $r \geq 3$. If r is even, then $N(G) = \mathbb{N}$, otherwise $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$. Furthermore, if r is odd and G is a 2-edge connected r -regular graph, then $N(G) = \mathbb{N} \setminus \{2\}$.*

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References

- [1] S. Akbari, F. Rahmati, S. Zare, Zero-sum magic labelings and null sets of regular graphs, Electron. J. Combin. 21(2), 2014, # P2.17.
- [2] Q. R. Yu and G. Liu, Graph factors and matching extensions, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2009: 21-22.